Aggregate Implications of Micro Asset Market Segmentation: *Supplementary Online Appendix*

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This supplementary online appendix is organized as follows:

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A General model with detailed derivations

We add three features relative to the model presented in the main text: (i) for each market m there is a density $\omega_m \geq 0$ of traders, (ii) the asset supply is $S_m \geq 0$, not normalized to 1, and (iii) there are bonds in positive net supply held in the family portfolio. The total measure of traders is one:

$$\int_0^1 \omega_m \, dm = 1. \tag{1}$$

The average segmentation parameters is then taken to be $\bar{\lambda} := \int_0^1 \lambda_m \omega_m \, dm$. Each period one share of the asset produces a stochastic realization of a non-storable dividend $y_{m,t} > 0$. The aggregate endowment available to the entire economy is:

$$y_t = \int_0^1 y_{m,t} S_m \omega_m \, dm. \tag{2}$$

As in the text, traders in market m are assumed to bear an exogenous fraction $\lambda_m \in [0, 1]$ of the expense of purchasing assets in that market and in return receive λ_m of the benefit. The remaining $1 - \lambda_m$ of the expenses and the benefits is borne by the family. As show in the text, this results in a sequential budget constraint of the form:

$$c_{m,t} + \lambda_m p_{m,t} s_{m,t} + (1 - \lambda_m) p_{t,t}^F \leq \lambda_m (p_{m,t} + y_{m,t}) s_{m,t-1} + (1 - \lambda_m) \left(p_{t-1,t}^F + y_t \right) - \tau_{m,t}, \tag{3}$$

where the new term, $\tau_{m,t}$, is a lump-sum tax levied on market m by the government. As in the main text $p_{t-1,t}^F + y_t$ and $p_{t,t}^F$ represent the cum-dividend value of the family portfolio brought into the period and the ex-dividend value of the family portfolio acquired this period, respectively. Proceeding as in the text, we find that $p_{t,t}^F$ and $p_{t-1,t}^F$ satisfy:

$$\begin{aligned} (1 - \bar{\lambda}) \left(p_{t-1,t}^F + y_t \right) &= \int_0^1 (1 - \lambda_n)(p_{n,t} + y_{n,t}) s_{n,t-1} \omega_n \, dn + b_{1,t-1} + \sum_{k \geq 1} \pi_{k,t} b_{k+1,t-1} \\ (1 - \bar{\lambda}) p_{t,t}^F &= \int_0^1 (1 - \lambda_n) p_{n,t} s_{n,t} \omega_n \, dn + \sum_{k \geq 1} \pi_{k,t} b_{k,t}, \end{aligned}$$

where $\pi_{k,t}$ and $b_{k,t}$ denote the price and quantity of purchases of zero-coupon bonds that pay the family one (real) dollar for sure in k periods' time.

Government. The government collects lump-sum taxes from each market and issues zero-coupon bonds of various maturities subject to the period budget constraint:

$$B_{1,t-1} + \sum_{k\geq 1} \pi_{k,t} B_{k+1,t-1} \leq \sum_{k\geq 1} \pi_k B_{k,t} + \int_0^1 \tau_{m,t} \omega_m \, dm, \tag{4}$$

where $B_{k,t}$ denotes the government's issue of k-period bonds at time t. We choose a particular specification of lump-sum taxes that has the property of not redistributing resources across markets:

$$\tau_{m,t} = \frac{1-\lambda_m}{1-\bar{\lambda}} \left(B_{1,t-1} + \sum_{k\geq 1} \pi_{k,t} \left[B_{k+1,t-1} - B_{k,t} \right] \right). \tag{5}$$

Equilibrium allocations. Market clearing requires $s_{m,t} = S_m$ for each m and $b_{k,t} = B_{k,t}$ for each k. We plug these conditions in the market-specific budget constraints and then use the government budget constraint combined with the expressions (5) for lump-sum taxes. After cancelling common terms we get:

$$c_{m,t} = \lambda_m y_{m,t} S_m + (1-\lambda_m) \int_0^1 \frac{1-\lambda_n}{1-\bar{\lambda}} y_{n,t} S_n \omega_n \, dn.$$

First-order conditions and asset pricing. Let $\mu_{m,t} \geq 0$ denote the multiplier on the budget constraint for market m and use the market-specific budget constraints and accounting identities for the family portfolio to write the Lagrangian:

$$\mathscr{L} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \int_0^1 \left\{ u(c_{m,t}) + \mu_{m,t} \mathcal{B}_{m,t} \right\} \omega_m \, dm \right]$$

where

$$\begin{aligned}
\mathcal{B}_{m,t} =& \lambda_m (p_{m,t} + y_{m,t}) s_{m,t-1} \\
&+ \frac{1-\lambda_m}{1-\bar{\lambda}} \left(\int_0^1 (1-\lambda_n)(p_{n,t} + y_{n,t}) s_{n,t-1} \omega_n \, dn + b_{1,t-1} + \sum_{k\geq 1} \pi_{k,t} b_{k+1,t-1} \right) \\
&- \left[c_{m,t} + \lambda_m p_{m,t} s_{m,t} + \frac{1-\lambda_m}{1-\bar{\lambda}} \left(\int_0^1 (1-\lambda_n) p_{n,t} s_{n,t} \omega_n \, dn + \sum_{k\geq 1} \pi_{k,t} b_{k,t} \right) + \tau_{m,t} \right].
\end{aligned}$$

Now collecting terms in $\int_0^1 \mu_{m,t}\mathcal{B}_{m,t}\omega_m\, dm$ and rearranging:

$$
\begin{aligned}
&\int_0^1 \mu_{m,t}\mathcal{B}_{m,t}\omega_m\, dm \\
&= \int_0^1 \mu_{m,t}\Big\{\lambda_m(p_{m,t}+y_{m,t})s_{m,t-1} - c_{m,t} - \lambda_m p_{m,t}s_{m,t} - \tau_{m,t}\Big\}\omega_m\, dm \\
&+ \int_0^1 \mu_{m,t}\frac{1-\lambda_m}{1-\bar\lambda}\Big\{b_{1,t} + \sum_{k\geq 1}\pi_{k,t}(b_{k+1,t-1} - b_{k,t})\Big\}\,\omega_m\, dm \\
&+ \int_0^1 \mu_{m,t}\frac{1-\lambda_m}{1-\bar\lambda}\int_0^1(1-\lambda_n)\Big[(p_{n,t}+y_{n,t})s_{n,t-1} - p_{n,t}s_{n,t}\Big]\omega_n\omega_m\, dn\, dm.
\end{aligned}
$$

Now, in the last term, we permute the roles of the symbols m and n and then interchange the order of integration:

$$
\begin{aligned}
&\int_0^1 \mu_{n,t}\frac{1-\lambda_n}{1-\bar\lambda}\int_0^1(1-\lambda_m)[(p_{m,t}+y_{m,t})s_{m,t-1} - p_{m,t}s_{m,t}]\omega_m\omega_n\, dm\, dn \\
= \quad &\left[\int_0^1 \mu_{n,t}\frac{1-\lambda_n}{1-\bar\lambda}\omega_n\, dn\right]\int_0^1(1-\lambda_m)[(p_{m,t}+y_{m,t})s_{m,t-1} - p_{m,t}s_{m,t}]\omega_m\, dm.
\end{aligned}
$$

Next, define the weighted average of Lagrange multipliers:

$$
q_{m,t} := \lambda_m\mu_{m,t} + (1-\lambda_m)q_t, \text{ and } q_t := \int_0^1 \frac{1-\lambda_n}{1-\bar\lambda}\mu_{n,t}\omega_n\, dn,
$$

as in the main text. Substituting for $q_{m,t}$ and q_t we get:

$$
\begin{aligned}
\mathscr{L} = \mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\int_0^1 \Big\{&u(c_{m,t}) + q_{m,t}(p_{m,t}+y_{m,t})s_{m,t-1} - q_{m,t}p_{m,t}s_{m,t} \\
&- \mu_{m,t}(c_{m,t}+\tau_{m,t}) + q_t\Big(b_{1,t-1} + \sum_{k\geq 1}\pi_{k,t}(b_{k+1,t-1} - b_{k,t})\Big)\Big\}\omega_m\, dm\Bigg].
\end{aligned}
$$

Apart from the term reflecting the presence of bonds, this is the same Langrangian as in the main text. We take derivatives (point-wise) to obtain the first order necessary conditions reported in the main text.

Portfolio weights and returns. To streamline the exposition we return to the model used in the main text. The total value of the family portfolio is:

$$
\int_0^1 \frac{1-\lambda_m}{1-\bar\lambda}p_{m,t}s_{m,t}\, dm.
$$

Thus, in the family portfolio, asset m is represented with a weight:

$$\psi_{m,t} := \frac{\frac{1-\lambda_m}{1-\bar{\lambda}} p_{m,t} s_{m,t}}{\int_0^1 \frac{1-\lambda_n}{1-\bar{\lambda}} p_{n,t} s_{n,t} \, dn}.$$

Letting $R_{m,t+1} = (p_{m,t+1} + y_{m,t+1})/p_{m,t}$ be the return on asset m, the return on the family portfolio can be written:

$$R_{t+1} = \int_0^1 R_{m,t+1} \psi_{m,t} \, dm.$$

Now recall that trader m holds $\lambda_m p_{m,t} s_{m,t}$ real dollars of asset m, and the rest of his investment:

$$(1-\lambda_m) \int_0^1 \frac{1-\lambda_n}{1-\bar{\lambda}} p_{n,t} s_{n,t} \, dn,$$

is in the family portfolio. Thus, the return of trader's m portfolio can be written:

$$\Psi_{m,t} R_{m,t+1} + (1 - \Psi_{m,t}) R_{t+1},$$

where:

$$\Psi_{m,t} := \frac{\lambda_m p_{m,t} s_{m,t+1}}{\lambda_m p_{m,t} s_{m,t} + (1-\lambda_m) \int_0^1 \frac{1-\lambda_n}{1-\bar{\lambda}} p_{n,t} s_{n,t} \, dn},$$

is the portfolio weight in the local asset.

B Computational details

Information. The aggregate state is a VAR for log consumption growth and log idiosyncratic volatility:

$$
\begin{aligned}
\log g_{t+1} &= (1-\rho)\log\bar{g} + \rho\log g_t + \varepsilon_{g,t+1} \\
\log\sigma_{t+1} &= (1-\phi)\log\bar{\sigma} + \phi\log\sigma_t - \eta\left(\log g_t - \log\bar{g}\right) + \varepsilon_{v,t+1},
\end{aligned}
$$

where $0 \leq \rho, \phi < 1$ and where the two components of innovation, $\epsilon_{g,t+1}$ and $\epsilon_{v,t+1}$, are assumed to be contemporaneously uncorrelated. The dividend in market m is:

$$
\log y_{m,t} = \log y_t + \log\hat{y}_{m,t}, \tag{6}
$$

where the log idiosyncratic component is conditionally IID normal in the cross section:

$$
\begin{aligned}
\log\hat{y}_{m,t} &\sim \text{ IID across } m \text{ and } N(-\sigma_{mt}^2/2, \sigma_{mt}^2) \\
\sigma_{mt} &= \sigma_t\hat{\sigma}_m,
\end{aligned}
$$

for some time-invariant market specific volatility level $\hat{\sigma}_m$.

Setup. Let utility be CRRA with coefficient $\gamma > 0$ so $u'(c) = c^{-\gamma}$. Assume markets come in M different *types* $m \in \{1, \ldots, M\}$. Note that this is an abuse of notation given that we previously used m to index a single market within the $[0,1]$ continuum. There is an equal measure of assets, $1/M$, in each market type. The total measure of traders in a market of type m is denoted by ω_m. Thus, we have the restriction:

$$
\sum_{m=1}^{M}\omega_m = 1.
$$

The supply of asset per trader in a market of type m is S_m, so the total supply in that market is $S_m\omega_m$. The dividend is $y_{m,t} = y_t\hat{y}_{m,t}$ where $\mathbb{E}\left[\hat{y}_{m,t} \,|\, g_t, \sigma_t\right] = 1$. Since the aggregate endowment is y_t, we need to impose the restriction:

$$
\sum_{m=1}^{M} S_m\omega_m = 1.
$$

The segmentation parameter in a market of type m is λ_m and the supply per trader is S_m. In equilibrium, consumption in a market of type m is given by:

$$c_{m,t} = y_t \left(A_m + B_m \hat{y}_{m,t} \right),$$

where

$$A_m := (1 - \lambda_m) \sum_{n=1}^{M} \frac{1 - \lambda_n}{1 - \bar{\lambda}} S_n \omega_n,$$

and

$$B_m := \lambda_m S_m.$$

We then have $q_{m,t} = \theta_{m,t} y_t^{-\gamma}$ where:

$$\theta_{m,t} = \lambda_m \left(A_m + B_m \hat{y}_{m,t} \right)^{-\gamma} + (1 - \lambda_m) \sum_{n=1}^{M} \frac{1 - \lambda_n}{1 - \bar{\lambda}} \mathbb{E} \left[\left(A_n + B_n \hat{y}_{n,t} \right)^{-\gamma} \mid g_t, \sigma_t \right] \omega_n,$$

where, by the LLN, the conditional expectation on the right–hand side calculates the cross-sectional average of $(A_n + B_n \hat{y}_{n,t})^{-\gamma}$ within type n markets. We explain below how to compute this expectation. Now let $\hat{p}_{m,t} := p_{m,t}/y_t$ be the price/dividend ratio in a type m market. This solves:

$$\hat{p}_{m,t} = \mathbb{E}_t \left[\beta g_{t+1}^{1-\gamma} \frac{\theta_{m,t+1}}{\theta_{m,t}} (\hat{p}_{m,t+1} + \hat{y}_{m,t+1}) \right]. \tag{7}$$

B.1 Approximation

Each market is characterized by 3 states: two aggregate states (g, σ) and one idiosyncratic state $\hat{y}_m$ (to simplify notation, we omit the 'log'). Given the specification above, the transition density is of the form:

$$f(g', \sigma', \hat{y}' \mid g, \sigma, \hat{y}) = f(g', \sigma' \mid g, \sigma) f(\hat{y}' \mid \sigma').$$

Our approximation follows Tauchen and Hussey (1991). First, we pick quadrature nodes and weights for the aggregate state: consumption growth, Q_g and W_g (column vectors of size N_g) and volatility, Q_σ and W_σ (column vectors of size N_σ).

In their original paper, Tauchen and Hussey recommended to pick these nodes and weights according to the transition density evaluated at the mean, i.e., a bivariate Gaussian density $f(g', \sigma' \mid \bar{g}, \bar{\sigma})$

which in the present case is the product of two independent normal densities with means $\log \bar{g}, \log \bar{\sigma}$, respectively, and variances σ_g^2 and σ_v^2. Subsequent work has highlighted, however, that when the Markov chain being approximated is highly persistent, the quality of the approximation may be poor. In our calibration exercise, this problem may arises when the moment matching algorithm searches in the region where the volatility process, σ, is highly persistent (ϕ close to 1). To alleviate this concern we follow Flodèn (2008): we generate nodes and weights for σ based on a "twisted" Gaussian density with a higher standard deviation:

$$\sigma = w\sigma_v + (1-w)\frac{\sigma_v}{\sqrt{1-\phi^2}} \qquad \text{where } w = 1/2 + \phi/4. \tag{8}$$

We also use a larger number of nodes to better capture the impact of high realization of σ. Below, we provide further discussion of the robustness of the approximation.

Next, for every quadrature value of σ, we generate quadrature nodes and weights in each market type m for the log idiosyncratic state $\log \hat{y}$, according to a Gaussian density with mean $-\hat{\sigma}_m^2\sigma^2/2$ and variance $\hat{\sigma}_m^2\sigma^2$. The resulting nodes and weights column vectors have length $N_\sigma \times N_{\hat{y}}$ and we denote them by $Q^m_{\hat{y}\,|\,\sigma}$ and $W^m_{\hat{y}\,|\,\sigma}$. In these vectors of nodes and weights, we adopt the convention that "idiosyncratic endowment comes first:" that is, in the quadrature node vector, idiosyncratic endowment i under volatility j is found in entry $i + N_{\hat{y}}(j-1)$.

Now, if we combine idiosyncratic endowment, aggregate volatility, and aggregate endowment growth together we obtain, for each market type m, a finite state space that we index by $n \in \{1, 2, 3, \ldots N\}$, where

$$N \equiv N_{\hat{y}} \times N_\sigma \times N_g.$$

We adopt the convention the state of idiosyncratic endowment $i \in \{1, \ldots, N_{\hat{y}}\}$, volatility $j \in \{1, \ldots, N_\sigma\}$, and aggregate consumption growth $k \in \{1, \ldots, N_g\}$ correspond to state:

$$n = i + N_{\hat{y}}(j-1) + N_{\hat{y}}N_\sigma(k-1).$$

In each state, the value of idiosyncratic endowment, aggregate volatility, and aggregate consumption growth can be conveniently represented with Kronecker products of the quadrature nodes:

$$\begin{aligned} V_g &= Q_g \otimes e_{N_\sigma} \otimes e_{N_y} \\ V_\sigma &= e_{N_g} \otimes Q_\sigma \otimes e_{N_y} \\ V^m_{\hat{y}} &= e_{N_g} \otimes Q^m_{\hat{y}\,|\,\sigma}, \end{aligned}$$

where e_N denotes a $N \times 1$ vector of ones. By construction, entry n of vector V_g contains consumption growth if the state of market m is n, and similarly for V_σ and $V_{\hat{y}}^m$. The corresponding quadrature weights are obtained as follows. We let:

$$\begin{aligned} A &= W_g \otimes e_{N_\sigma} \otimes e_{N_y} \\ B &= e_{N_g} \otimes W_\sigma \otimes e_{N_y} \\ C^m &= e_{N_g} \otimes W_{\hat{y}\,|\,\sigma}^m, \end{aligned}$$

so that the quadrature weights for the state are:

$$W^m = A. * B. * C^m$$

where $.*$ denotes Matlab coordinate-per-coordinate product.

Transition probability matrix. To implement the method of Tauchen and Hussey (1991), we define a Matlab function:

$$f^m(s' \,|\, s) = f^m(\hat{y}' \,|\, \sigma') \times f(\sigma' \,|\, \sigma, g) \times f(g' \,|\, g),$$

as well as the quadrature weighting function:

$$\omega^m(s) = \omega^m(\hat{y} \,|\, \sigma) \times \omega(\sigma) \times \omega(g),$$

which is the probability density function used above to generate the quadrature nodes and weights for market m. Letting , the matrix formula for the transition matrix is:

$$\begin{aligned} G \;=\; & f^m(e_N\, V_{\hat{y}}' \,|\, e_N\, V_\sigma'). * f(e_N\, V_\sigma' \,|\, V_\sigma\, e_N', V_g\, e_N'). * f(e_N\, V_g' \,|\, V_g\, e_N') \\ & . * (e_N * W')./\left[e_N. * \omega(V_{\hat{y}}' \,|\, V_\sigma'). * \omega(V_\sigma'). * \omega(V_g')\right], \end{aligned}$$

which we then normalize so that the rows sum to 1.

Calculating cross-sectional moments. In many instance in the program we need to calculate

$$\mathbb{E}\left[x_m \,|\, g, \sigma\right],$$

for some random variable x_m. To do this, we consider:

$$K_\sigma = (I_{N_g \times N_\sigma} \otimes e_{N_{\hat{y}}}')\;[x_m. * W^m],$$

where

$$W^m = e_{N_g} \otimes W^m_{\hat{y}\,|\,\sigma}.$$

The coordinate-wise product multiplies each realization of x_m by its probability conditional on (g, σ), and the pre-multiplication adds up. We then re-Kroneckerize this in order to obtain a $N \times 1$ vector:

$$K_\sigma \otimes e_{N_{\hat{y}}}.$$

B.2 Robustness of the approximation

Table II shows that our numerical results are robust to alternative parameterizations of the numerical approximations. We consider three versions of the single λ economy: the benchmark version, the version with constant σ, and the feedback version with countercyclical σ_t. In our default *standard* parameterization we have $N = N_g \times N_\sigma \times N_{\hat{y}} = 3 \times 9 \times 19 = 513$ quadrature nodes and weights. It also uses the "twisted" density recommended by Flodèn (2008) to alleviate concerns about the accuracy of the Tauchen and Hussey (1991) procedure when the σ_t process is persistent (see equation (8) above). In our *high* precision parameterization we have $N = N_g \times N_\sigma \times N_{\hat{y}} = 5 \times 19 \times 25 = 2,375$ nodes and weights and again use the twisting recommended by Flodèn. In the *no twist* parameterization we use the plain Tauchen and Hussey (1991) procedure and the same configuration of nodes as in the standard parameterization. The issue of twisting does not arise in the constant σ model.

For each of these numerical approximations the table reports the calibrated parameter values, the values of the moments we target, and the implications for aggregate asset prices.

For a given model, we see that increasing the number of nodes from the standard to high parameterization has negligible effect on the results. Similarly, the twisting recommended by Flodèn has negligible effect. This suggests that our calibrated stochastic process is not persistent enough to cause any problems for the plain Tauchen and Hussey procedure.

C Incomplete markets counterpart

In this Appendix we consider an incomplete markets counterpart of our model. In contrast with the segmented markets model, we assume that traders are only restricted in their local trades, i.e., traders in market $m \in [0, 1]$ have to hold at least λ shares of their local assets. As shown in detail below, we solve for an equilibrium in two steps. First, we consider an *alternate* model where traders faces tighter constraints and are restricted to a smaller set of securities. Namely, we start by assuming that trader $m \in [0, 1]$ is forced to hold exactly λ shares of asset m, and can only trade a claim to aggregate consumption, that is, a well diversified portfolio of assets $n \neq m$. This becomes a simple Bewley model whose equilibrium can be characterized using results from Krueger and Lustig (2010). Second, we show that the prices and allocations in this alternate model are the basis of an equilibrium in the original incomplete markets model. Specifically:

- the *ex-dividend* price of any local asset is the same as the price of a claim to aggregate consumption,
- trader $m \in [0, 1]$ always finds it optimal to hold a well diversified portfolio of assets $n \neq m$, and
- the portfolio constraint of trader $m \in [0, 1]$ is binding. That is, if we allow a trader to hold more than λ shares, her optimal holding remains equal to λ.

The intuition for these results is the following. Given that all traders $n \neq m$ can trade asset m without portfolio constraints, their marginal rate of substitution (MRS) must price asset m. Moreover, the MRS of traders $n \neq m$ only depends on the history of dividends in market $n \neq m$, not on the history of dividends in market m. Therefore, from the point of view of traders $n \neq m$, the dividend risk in market m is idiosyncratic. It follows that the price of asset m must be the same as the price of a claim to aggregate consumption. Given that all assets have the same ex-dividend price, trader m wants to hold a well diversified equally-weighted portfolio of assets $n \neq m$ and wants to hold as little of asset m as possible, i.e., exactly λ shares.

C.1 Alternate model

We assume that the aggregate endowment, y_t, follows a geometric random walk:

$$y_t = g_t y_{t-1}$$

where y_0 is given and where g_t is IID with finite support $\mathcal{G}$. We also assume that there is a continuum $m \in [0,1]$ of assets with dividends $\hat{y}_{m,t} y_t$, where $\hat{y}_{m,t}$ is IID across time and assets, has finite support $\mathcal{Y}$, and is independent from the endowment growth process. The mean of $\hat{y}_{m,t}$ is normalized to one. There is a continuum of traders, also indexed by $m \in [0,1]$.

We consider a version of the incomplete markets model of Krueger and Lustig (2010): we assume that a trader of type $m \in [0,1]$ is forced to hold λ shares of asset m but can self-insure by trading claims to the aggregate endowment.[1]

The initial distribution of aggregate consumption claim holdings is $\Phi_0(\sigma)$, with $\int \sigma d\Phi_0(\sigma) = 1 - \lambda$. Now consider an individual trader who starts with initial holding σ_0. At time $t \geq 1$ after history $s_m^t = (\hat{y}_m^t, g^t) := (\hat{y}_{m,1}, \ldots, \hat{y}_{m,t}, g_1, \ldots, g_t)$, the trader chooses consumption $c_t(\sigma_0, s_m^t)$ and asset holdings $\sigma_t(\sigma_0, s_m^t)$, subject to the sequential budget constraint:

$$c_t(\sigma_0, s_m^t) + \sigma_t(\sigma_0, s_m^t) p_t(g^t) \leq \lambda \hat{y}_{m,t} y_t + \sigma_{t-1}(\sigma_0, s_m^{t-1})[y_t + p_t(g^t)], \tag{9}$$

where $p_t(g^t)$ is the price of a consumption claim after aggregate history g^t. On the right-hand side of the budget constraint, $\lambda \hat{y}_{m,t} y_t$ represents the dividend paid out by the λ shares of asset m that the trader is forced to hold. We also assume that the trader faces short-selling limits of the sort considered in Krueger and Lustig:

$$\sigma_t(\sigma_0, s_m^t) p_t(g^t) \geq -K_t y_t. \tag{10}$$

Intertemporal utility is

$$\sum_{t=1}^{\infty} \sum_{s_m^t} \beta^t \pi_t(s_m^t) \frac{c_t(\sigma_0, s_m^t)^{1-\gamma}}{1-\gamma}, \tag{11}$$

where $\pi_t(s_m^t)$ denotes the probability of history s_m^t. An *equilibrium* consists of asset prices $\{p_t(g^t)\}$ and policy functions $\{c_t(\sigma_0, s_m^t)\}$ and $\{\sigma_t(\sigma_0, s_m^t)\}$ such that the policy functions maximize each

[1] Krueger and Lustig (2010) also consider richer market structures, with Arrow securities paying off conditional on the realized aggregate state, and one-period riskless bonds. However, they show that there are equilibria in which there is no trade in these other markets. That is, in order to self-insure against idiosyncratic shocks, agents find it optimal to trade only aggregate endowment claims.

trader's problem given prices, and markets clear at for all t and g^t:

$$\int \sum_{\hat{y}_m^t} \pi_t(\hat{y}_m^t) c_t(\sigma_0, \hat{y}_m^t, g^t) d\Phi(\sigma_0) = y_t,$$
$$\int \sum_{\hat{y}_m^t} \pi_t(\hat{y}_m^t) \sigma_t(\sigma_0, \hat{y}_m^t, g^t) d\Phi(\sigma_0) = 1 - \lambda.$$

A rescaled economy. To solve for an equilibrium, Krueger and Lustig (2010) start with the following change of variables:

$$\hat{c}_t(\sigma_0, s_m^t) := \frac{c_t(\sigma_0, s_m^t)}{y_t}, \quad \hat{\sigma}_t(\sigma_0, s_m^t) := \sigma_t(\sigma_0, s_m^t), \quad \text{and} \quad \hat{p}_t(g^t) := \frac{p_t(g^t)}{y_t}.$$

With this new notation, a trader's intertemporal utility can be written:

$$y_0^{1-\gamma} \sum_{t=1}^{\infty} \hat{\beta}^t \sum_{s_m^t} \hat{\pi}_t(s_m^t) \frac{\hat{c}_t(\sigma_0, s_m^t)^{1-\gamma}}{1-\gamma},$$
$$\text{where} \quad \hat{\beta} := \beta \sum_{g \in \mathcal{G}} \pi(g) g^{1-\gamma}, \quad \text{and} \quad \hat{\pi}_t(s_m^t) := \pi_t(\hat{y}_m^t) \prod_{s=1}^{t} \frac{\pi(g_s) g_s^{1-\gamma}}{\sum_{g \in \mathcal{G}} \pi(g) g^{1-\gamma}}.$$

Similarly, the sequential budget constraints and the short-selling constraints now become:

$$\hat{c}_t(\sigma_0, s_m^t) + \hat{\sigma}_t(\sigma_0, s_m^t) \hat{p}_t(g^t) \leq \lambda \hat{y}_{m,t} + \hat{\sigma}_{t-1}(\sigma_0, s_m^{t-1})[1 + \hat{p}_t(g^t)]$$
$$\hat{\sigma}_t(\sigma_0, s_m^t) \hat{p}_t(g^t) \geq -K_t,$$

with market clearing conditions:

$$\int \sum_{y_m^t} \pi_t(\hat{y}_m^t) \hat{c}_t(\sigma_0, s_m^t) d\Phi(\sigma_0) = 1$$
$$\int \sum_{y_m^t} \pi_t(\hat{y}_m^t) \hat{\sigma}_t(\sigma_0, s_m^t) d\Phi(\sigma_0) = 1 - \lambda.$$

An equilibrium of the rescaled economy is defined exactly as before.

As is clear from these equations, after the change of variables, the history of aggregate endowment growth g^t no longer affects the fundamentals of the rescaled economy. Indeed, y_t does not affect the right–hand side of the rescaled market clearing conditions, and the only way it affects the agent's budget constraints is through its potential impact on the rescaled asset price, $\hat{p}_t(g^t)$. It is therefore natural to look for an equilibrium in which the rescaled asset price is, in fact, a deterministic function of time, i.e. $\hat{p}_t(g^t) = \hat{p}_t$, and in which rescaled consumption and asset holdings are only functions of time and of the history of idiosyncratic shocks, $\hat{y}_m^t$, i.e., $\hat{c}_t(\sigma_0, s_m^t) = \hat{c}_t(\sigma_0, \hat{y}_m^t)$, and $\sigma_t(\sigma_0, s_m^t) = \sigma_t(\sigma_0, \hat{y}_m^t)$. In this case, the asset becomes a risk-free bond and an equilibrium can be computed using standard methods for Bewley models (Ljungqvist and Sargent, 2004, Chapter 17, for example).

After solving for an equilibrium of the rescaled economy, an equilibrium of the incomplete markets model is found by scaling back the price, consumption, and asset holdings:

$$p_t(g^t) = y_t \hat{p}_t, \quad c_t(\sigma_0, \hat{y}_m^t, g^t) = y_t \hat{c}_t(\sigma_0, \hat{y}_m^t), \quad \text{and} \quad \sigma_t(\sigma_0, \hat{y}_m^t, g^t) = \hat{\sigma}_t(\sigma_0, \hat{y}_m^t).$$

C.2 Back to the original incomplete markets model

With this result in mind, we provide an equilibrium in the original incomplete markets model, i.e., where each trader $m \in [0,1]$ can trade claims in all assets but is restricted to hold at least λ shares of their local asset. The trader faces the short-selling restriction that the total value of her portfolio has to be greater than $-K_t y_t + \lambda p_{m,t}$, where $p_{m,t}$ is the price of the local asset. We guess and verify that there exists an equilibrium in which:

- all local assets have the same price $p_{m,t} = p_t(g^t) = \hat{p}_t y_t$,
- the trader's consumption is the same as in the alternative incomplete market model,
- trader m holds λ shares of asset m and $\hat{\sigma}_t(\hat{y}_m^t)$ shares of a claim to the aggregate endowment. The trader synthesizes this claim by holding an equally weighted portfolio of assets $n \neq m$.

The asset market clears by construction. Also by construction, the sequential budget constraints and the short-selling restrictions hold. So, all we need to verify is that the consumption and asset holdings are individually optimal.

Optimality of holdings of asset $n \neq m$. Given concavity, the first-order conditions are necessary and sufficient. The first-order condition for the holdings of asset $n \neq m$ is:

$$\hat{p}_t y_t = \beta \sum_{s_{m,t+1}, \hat{y}_{n,t+1}} \pi(g_{t+1}) \pi(\hat{y}_{m,t+1}) \pi(\hat{y}_{n,t+1}) \left(\frac{y_{t+1} \hat{c}_{t+1}(\sigma_0, \hat{y}_m^{t+1})}{y_t \hat{c}_t(\sigma_0, \hat{y}_m^t)} \right)^{-\gamma} [y_{t+1} \hat{y}_{n,t+1} + \hat{p}_{t+1} y_{t+1}] + \nu_{m,t},$$

with $\nu_{m,t} \geq 0$, and $\nu_{m,t} = 0$ if the short-selling restriction is slack. Note that, in this first–order condition, we used the fact that, in our candidate equilibrium, re-scaled consumption does not depend on the history of aggregate shocks. Dividing both sides by $y_t > 0$, and keeping in mind that

$y_{t+1}/y_t = g_{t+1}$, we can rewrite this condition as:

$$
\begin{aligned}
\hat{p}_t &= \hat{\beta} \sum_{g_{m,t+1}, \hat{y}_{m,t+1}, \hat{y}_{n,t+1}} \hat{\pi}(g_{t+1})\pi(\hat{y}_{m,t+1})\pi(\hat{y}_{n,t+1}) \left(\frac{\hat{c}_{t+1}(\sigma_0, \hat{y}_m^{t+1})}{\hat{c}_t(\sigma_0, \hat{y}_m^t)} \right)^{-\gamma} [\hat{y}_{n,t+1} + \hat{p}_{t+1}] + \frac{\nu_{m,t}}{y_t} \\
&= \hat{\beta} \sum_{g_{t+1}} \hat{\pi}(g_{t+1}) \sum_{\hat{y}_{m,t+1}} \pi(\hat{y}_{m,t+1}) \left(\frac{\hat{c}_{t+1}(\sigma_0, \hat{y}_m^{t+1})}{\hat{c}_t(\sigma_0, \hat{y}_m^t)} \right)^{-\gamma} \left[\sum_{\hat{y}_{n,t+1}} \pi(\hat{y}_{n,t+1})\hat{y}_{n,t+1} + \hat{p}_{t+1} \right] + \frac{\nu_{m,t}}{y_t} \\
&= \hat{\beta} \sum_{\hat{y}_{n,t+1}} \pi(\hat{y}_{m,t+1}) \left(\frac{\hat{c}_{t+1}(\sigma_0, \hat{y}_m^{t+1})}{\hat{c}_t(\sigma_0, \hat{y}_m^t)} \right)^{-\gamma} [1 + \hat{p}_{t+1}] + \frac{\nu_{m,t}}{y_t} \qquad (12)
\end{aligned}
$$

where we use that $\hat{y}_{m,t+1}$ and $\hat{y}_{n,t+1}$ are independent, that $\sum_{g_{t+1}} \hat{\pi}(g_{t+1}) = 1$, and finally that $\sum_{\hat{y}_{n,t+1}} \pi(\hat{y}_{n,t+1})\hat{y}_{n,t+1} = 1$. This condition is the same as the one for the aggregate consumption claim in the alternative incomplete markets model. It thus holds by construction. The key intuition is that, for agent m, the endowment risk of asset $n \neq m$ is idiosyncratic. Therefore, this agent values a claim to asset $n \neq m$ exactly the same way as a claim to aggregate endowment.

Optimality of holding of asset m. For agent $m \in [0, 1]$, the first-order condition for the holding of asset m is:

$$
\hat{p}_t y_t \geq \sum_{g_{t+1}, \hat{y}_{m,t+1}} \beta\pi(g_{t+1})\pi(\hat{y}_{m,t+1}) \left(\frac{y_{t+1}\hat{c}_{t+1}(\hat{y}_m^{t+1})}{y_t \hat{c}_t(\hat{y}_m^t)} \right)^{-\gamma} [y_{t+1}\hat{y}_{m,t+1} + \hat{p}_{t+1}y_{t+1}] + \nu_{m,t}.
$$

where $\nu_{m,t}$ is defined as above. We need to verify an inequality because of the restriction that agent m has to hold at least λ shares of asset m, and because of our guess that the agent holds exactly λ shares. Proceeding as above we can rewrite this condition as:

$$
\hat{p}_t \geq \sum_{\hat{y}_{m,t+1}} \hat{\beta}\pi(\hat{y}_{m,t+1}) \left(\frac{\hat{c}_{t+1}(\sigma_0, \hat{y}_m^{t+1})}{\hat{c}_t(\sigma_0, \hat{y}_m^t)} \right)^{-\gamma} [\hat{y}_{m,t+1} + \hat{p}_{t+1}] + \frac{\nu_{m,t}}{y_t}.
$$

Substituting (12) on the left-hand side of this inequality, this condition becomes:

$$
\begin{aligned}
& \sum_{\hat{y}_{m,t+1}} \pi(\hat{y}_{m,t+1}) \left(\frac{\hat{c}_{t+1}(\sigma_0, \hat{y}_m^{t+1})}{\hat{c}_t(\sigma_0, \hat{y}_m^t)} \right)^{-\gamma} [\hat{y}_{m,t+1} - 1] \leq 0 \\
\Leftrightarrow \quad & \mathrm{Cov}_t \left[\left(\hat{c}_{t+1} \left(\sigma_0, \hat{y}_m^{t+1} \right) \right)^{-\gamma}, \hat{y}_{m,t+1} \right] \leq 0.
\end{aligned}
$$

That is, the agent finds it optimal to hold exactly λ shares of the asset if the asset payoff is negatively correlated with their marginal utility of consumption. This happens if, conditional on history $\hat{y}_m^t$, consumption next period is an increasing function of the local endowment realization, $\hat{y}_{m,t+1}$. But this follows from a known property of Bewley models: consumption is an increasing function of "cash-at-hand". In terms of our notation, this property can be expressed as follows:

Proposition. *Suppose that, for all σ_0 and $\hat{y}_m^t$, $\hat{c}_t(\sigma_0, \hat{y}_m^t) > 0$. Then $\hat{c}_t(\sigma_0, \hat{y}_m^t)$ is an increasing function of $\hat{y}_{m,t}$.*

Proof. Let

$$R_t := \frac{1 + \hat{p}_{t+1}}{p_t}$$

and consider the income fluctuation problem associated with the incomplete markets model. That is, for each $t \geq 1$, consider:

$$v_t(a) = \sup \sum_{j=0}^{\infty} \sum_{\hat{y}_m^{t+j} \succeq \hat{y}_m^t} \hat{\beta}^j \pi(\hat{y}_m^{t+j} \mid \hat{y}_m^t) \frac{c_{t+j}(\hat{y}_m^{t+j})^{1-\gamma}}{1-\gamma},$$

subject to

$$\begin{aligned} c_{t+j}(\hat{y}_m^j) + \frac{b_{t+j}(\hat{y_m}^{t+j})}{R_{t+j}} &\leq a_{t+j}(\hat{y}_m^{t+j}) \\ a_{t+j+1}(\hat{y}_m^{t+j+1}) &= \lambda \hat{y}_{m,t+j+1} + b_{t+j}(\hat{y}_m^{t+j}) \\ \frac{b_{t+j}(\hat{y}_m^{t+j})}{R_{t+j}} &\geq K_{t+j} \\ c_{t+j}(\hat{y}_m^j) &\geq 0 \\ a_t &= a. \end{aligned}$$

Given that the idiosyncratic dividends are IID over time, the optimization problem and therefore the value function only depend on time, not on the history $\hat{y}_m^t$ of idiosyncratic shocks up to time t. Because the objective is concave and the constraint set convex, it follows that the value function $v_t(a)$ is concave. Moreover, following the proof of Theorem 4.2 in Stokey and Lucas (1989) we find that the value function solves the Bellman equation:

$$v_t(a) = \sup_{c \geq 0} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \hat{\beta} \sum_{\hat{y}'_m} \pi(\hat{y}'_m) v_{t+1}\Big(\lambda \hat{y}'_m + R_t \left[a - c\right]\Big) \right\},$$

subject to $a - c \geq K_t$. In particular, this implies that consumption $c_t(a) := \hat{c}(\sigma_0, \hat{y}_m^t) > 0$ solves the Bellman equation at time t given cash-at-hand:

$$a = \lambda \hat{y}_{m,t} + \sigma_{t-1}\left(\sigma_0, \hat{y}_m^{t-1}\right)[1 + \hat{p}_t].$$

We now show that the value function is differentiable at a with $v_t'(a) = c_t(a)^{-\gamma}$. The proof is standard. Given that $c_t(a) > 0$, for $\tilde{a}$ close enough to a, the consumption $\tilde{c} = c_t(a) + \tilde{a} - a$ is feasible given cash-at-hand $\tilde{a}$ (it is positive and satisfies the borrowing constraint by construction). Plugging this back into the Bellman equation we obtain:

$$\begin{aligned} v_t(\tilde{a}) &\geq \frac{(c_t(a) + \tilde{a} - a)^{1-\gamma}}{1-\gamma} + \hat{\beta} \sum_{\hat{y}_m'} \pi(\hat{y}_m') v_{t+1}\left(\lambda \hat{y}_m' + R_t\left[a - c_t(a)\right]\right) \\ &= \frac{(c_t(a) + \tilde{a} - a)^{1-\gamma}}{1-\gamma} + v_t(a) - \frac{c_t(a)^{1-\gamma}}{1-\gamma}. \end{aligned}$$

Rearranging gives:

$$v_t(\tilde{a}) - v_t(a) \geq \frac{(c_t(a) + \tilde{a} - a)^{1-\gamma}}{1-\gamma} - \frac{c_t(a)^{1-\gamma}}{1-\gamma}.$$

Now consider $\tilde{a} > a$, divide both sides by $\tilde{a} - a > 0$, and let $\tilde{a} \to a^+$. Given that the function $v_t(a)$ is concave, it has left- and right-hand side derivatives everywhere. Therefore, as $\tilde{a} \to a^+$, the left-hand side of the above equation converges to the right derivative of the value function at a, so that we obtain:

$$v_t'(a^+) \geq c_t(a)^{-\gamma}.$$

Now do the same for $\tilde{a} < a$ and obtain:

$$v_t'(a^-) \leq c_t(a)^{-\gamma}.$$

Concavity also implies that $v_t'(a^-) \geq v_t'(a^+)$. Taken together, we find that $v_t(a)$ is differentiable at a and that $v_t'(a) = c_t(a)^{-\gamma}$. Using the notation of the sequence problem, this can be written:

$$c_t(\sigma_0, y_m^t) = \left[v_t'\left(\lambda \hat{y}_{m,t} + \sigma_{t-1}\left(\sigma_0, \hat{y}_m^{t-1}\right)[1 + \hat{p}_t]\right)\right]^{-\frac{1}{\gamma}}.$$

By concavity, the directional derivative of $v_t(a)$ is a decreasing functions of cash-at-hand. Together with the above, this implies that consumption is an increasing function of the current dividend realization, $\hat{y}_{m,t}$. $\square$

C.3 Different asset pricing implications

Another important difference between incomplete and segmented markets concerns the relationship between idiosyncratic income risk and the equity premium. As emphasized by Mankiw (1986), Constantinides and Duffie (1996) and Krueger and Lustig (2010), with CRRA utility and idiosyncratic income risk that is independent of aggregate consumption growth, idiosyncratic risk has *no* impact on the equity premium in the incomplete markets model.[2] Indeed, as explained above, in the incomplete markets version of our model, the MRS of *every* trader m prices the excess returns in market $n \neq m$. In particular, it prices the excess return of the market portfolio:

$$\mathbb{E}\left[M_m R^e\right] = 0. \tag{13}$$

Moreover, the MRS can be factored into $\hat{M}_m M$, where $M = \beta g^{-\gamma}$ is the Lucas-Breeden stochastic discount factor, and $\hat{M}_m$ is an idiosyncratic component that is independent from M. Expanding the expectation in (13) we have:

$$\mathbb{E}[\hat{M}_m M R^e] = \mathbb{E}[\hat{M}_m]\mathbb{E}[M R^e] + \mathrm{Cov}[\hat{M}_m, M R^e] = 0.$$

From independence $\mathrm{Cov}[\hat{M}_m, M R^e] = 0$. Using this and dividing by $\mathbb{E}[\hat{M}_m] > 0$ we obtain:

$$\mathbb{E}\left[M R^e\right] = 0.$$

As shown by Kocherlakota (1996), this asset pricing equation cannot rationalize the observed equity premium.

This irrelevance result does not hold in the segmented markets model. The reason is that in our asset pricing model the local stochastic discount factor does not have to price the excess return on the *aggregate* market portfolio, as in equation (13), but instead only has to price the excess return on the *local* asset market. The local discount factor is correlated with the local excess return (through the local endowment realization) and this makes it impossible to strip out the influence of the market-specific factor.

Specifically, instead of equation (13) we have a pricing equation of the form:

$$\mathbb{E}\left[M_m R^e_m\right] = 0, \tag{14}$$

[2] See Telmer (1993) and Heaton and Lucas (1996) for important early applications of incomplete markets models to asset pricing.

where M_m is the local stochastic discount factor and R^e_m is the local excess return. We can again write the local discount factor $M_m = \hat{M}_m M$ where M is the Lucas-Breeden discount factor and $\hat{M}_m$ is a market-specific factor. Now proceeding as above and expanding the expectation in (14) we have:

$$\mathbb{E}[\hat{M}_m M R^e_m] = \mathbb{E}[\hat{M}_m]\mathbb{E}[M R^e_m] + \mathrm{Cov}[\hat{M}_m, M R^e_m] = 0.$$

But $\hat{M}_m$ and R^e_m depend on the same local risk factor so $\mathrm{Cov}[\hat{M}_m, M R^e_m] \neq 0$ and we cannot strip out $\mathbb{E}[\hat{M}_m]$. This makes it impossible to aggregate the collection of equations (14) into (13), and, because of this, the standard incomplete markets logic does not apply in our model.

D Conditional moments and return predictability

Conditional price/dividend ratio. Our model's implications for time variation in asset returns are largely summarized by the implications for the market price/dividend ratio. The left panel of Figure I shows the annualized market price/dividend ratio as a function of the volatility state σ_t holding the aggregate endowment growth constant at its mean and using our benchmark parameterization unless otherwise noted. In the frictionless version of the model, $\lambda = 0$, the p_t/y_t ratio is constant. For $\lambda > 0$, the p_t/y_t ratio is monotonically declining. A high σ_t corresponds to high average marginal utility q_t and a low p_t/y_t and, in that sense, corresponds to a "bad" aggregate state. A low σ_t corresponds to a low average marginal utility q_t, a high p_t/y_t, and represents a "good" aggregate state. For higher values of λ, the price/dividend ratio is relatively lower in bad states and higher in good states. In short, more segmentation tends to amplify fluctuations in p_t/y_t.

Conditional volatility of stock returns. The right panel of Figure I shows the annualized conditional standard deviation of the market return. For $\lambda > 0$ this is monotonically increasing in the volatility state σ_t. An increase in σ_t represents an increase in the *cross-sectional* variation in idiosyncratic endowments, yet this translates to an increase in the *time series* variation of the aggregate market return. At high frequencies, the model produces ARCH-like effects in aggregate returns, the monthly autocorrelation coefficient for the conditional standard deviation of returns is 0.77. This would be undetectable in annual data ($0.77^{12} = 0.04$) but represents considerable time-variation in conditional return volatility at higher frequencies (on the order of $0.77^{1/30} = 0.99$ daily, say). Again we see that more segmentation tends to amplify fluctuations, here the sensitivity of the conditional standard deviation to σ_t is higher the higher is λ.

Return predictability. The time-variation in the price/dividend ratio shown in Figure I implies that aggregate market returns in our model are forecastable (given that aggregate endowment growth is IID). To see this, we use our model to reproduce return predictability regressions of the kind documented by Campbell and Shiller (1988) and Fama and French (1988). We run regressions of annual returns and excess returns on the dividend/price ratio y_t/p_t and a constant. In the data, at a one-year horizon this produces a coefficient on y_t/p_t of about 3 (for returns) or 3.4 (for excess returns). Thus, relatively low prices forecast high subsequent returns. In our model, we find the coefficient is about 15 for returns (Table III). Thus our model can reproduce the predictability of returns. However, because in our model the risk-free rate is nearly as countercyclical as returns, the model excess returns are nearly a-cyclical (in fact, the coefficient is slightly negative).

Another way to see the time-variation in returns is to observe that in the model the standard deviation of expected returns is 6.7%, just over two-thirds the level of the average return. In the data, the standard deviation of the fitted values of returns and excess returns are similarly volatile.

Since aggregate growth g_{t+1} is IID, the time-variation in asset returns in our model is introduced through the multiplicative adjustment θ_{t+1}/θ_t in the SDF that prices bonds and through the market-specific adjustments $\theta_{m,t+1}/\theta_{m,t}$ in the SDFs that price stocks. We now document the properties of these terms in more detail.

E Multiplicative adjustment to SDFs

Aggregate bond-pricing factor. With CRRA preferences the aggregate state price q_t can be written as the product of the marginal utility of aggregate consumption $y_t^{-\gamma}$ and a multiplicative term θ_t that captures the segmentation effect:

$$q_t = \theta_t y_t^{-\gamma},$$

where

$$\theta_t := \int_0^1 \frac{1-\lambda_m}{1-\bar{\lambda}}[1+\lambda_m(\hat{y}_{m,t}-1)]^{-\gamma}\, dm. \tag{15}$$

In other words, θ_t is the cross-sectional average marginal utility but reweighted to reflect the different contributions of traders in different markets to the family portfolio. Observe that θ_t depends on the *cross-sectional distribution* of endowments, as determined by the volatility factor σ_t, but does not depend on any individual endowment realization. A high realization of σ_t increases the cross-sectional dispersion of consumption $\hat{c}_{m,t} = 1+\lambda_m(\hat{y}_{m,t}-1)$ and, because $\hat{c}_{m,t}^{-\gamma}$ is convex, also increases θ_t.

The SDF that prices bonds is given by $\beta q_{t+1}/q_t$ so that the risk-free rate is:

$$R_{f,t} = \mathbb{E}_t\left[\beta g_{t+1}^{-\gamma}\frac{\theta_{t+1}}{\theta_t}\right]^{-1}.$$

Since g_{t+1} is IID, in the absence of time-variation in the multiplicative factor θ_{t+1}/θ_t, the risk-free rate $R_{f,t}$ would be constant. To understand the time-variation in the risk-free rate, Figure II illustrates how the conditional moments of θ_{t+1}/θ_t vary with σ_t for our model with a single λ. In this figure, we see that an increase in σ_t tends to reduce $\mathbb{E}_t[\theta_{t+1}/\theta_t]$. This is because while an increase in σ_t increases θ_t, mean-reversion implies that σ_{t+1} is not expected to be as high next period. Consequently, θ_{t+1} is not expected to be as high as θ_t. In short, when σ_t is relatively high θ_{t+1}/θ_t is expected to be low and the risk-free rate is high.

We report the quantitative properties of θ_{t+1}/θ_t in Table IV. For our benchmark calibration, we find θ_{t+1}/θ_t is on average 1.001, implying an annual growth rate of about 1.2% (i.e., this is approximately the amount by which the segmentation effects lower the risk-free rate relative to the frictionless benchmark) with a standard deviation of about 4.6% monthly, which is why the risk-free rate in our model is excessively volatile. Since θ_t is persistent but not a random walk, we find that θ_{t+1}/θ_t has a negative autocorrelation coefficient, -0.13 monthly. This gives rise to our model's upward-sloping average yield curve (as shown in Figure III).

Market-specific factors. Similarly, the state price in market m can be written:

$$q_{m,t} = \theta_{m,t} y_t^{-\gamma},$$

where

$$\theta_{m,t} := \lambda_m [1 + \lambda_m (\hat{y}_{m,t} - 1)]^{-\gamma} + (1 - \lambda_m)\theta_t,$$

and where θ_t is the aggregate adjustment given in (15) above. The market-specific SDF is then:

$$\beta \frac{q_{m,t+1}}{q_{m,t}} = \beta g_{t+1}^{-\gamma} \frac{\theta_{m,t+1}}{\theta_{m,t}}$$

Aggregate growth g_{t+1} enters only through the Lucas-Breeden factor $\beta g_{t+1}^{-\gamma}$; volatility σ_t enters only through the aggregate adjustments.

We report the quantitative properties of $\theta_{m,t+1}/\theta_{m,t}$ in Table IV. For our benchmark calibration, we find $\theta_{m,t+1}/\theta_{m,t}$ is on average 1.4% monthly and is very volatile, with a standard deviation of about 17% monthly. The only persistence in $\theta_{m,t}$ comes from σ_t through the aggregate θ_t. Consequently, the market-specific $\theta_{m,t}$ is less persistent than the aggregate θ_t. In turn, this implies that $\theta_{m,t+1}/\theta_{m,t}$ is *more* negatively serially correlated than the aggregate θ_{t+1}/θ_t, a monthly autocorrelation coefficient of -0.46 as opposed to -0.13.

Our model's implications for risk-premia depend also on the correlation of this multiplicative factor with the local endowment. The correlation of $\theta_{m,t+1}/\theta_{m,t}$ with $\hat{y}_{m,t+1}/\hat{y}_{m,t}$ is indeed quite negative, -0.76. The fluctuations in the σ_t impart some serial correlation to the conditional standard deviation of the market-specific SDF, about 0.58 monthly. This would not be detectable in annual data ($0.58^{12} = 0.001$) but represents considerable time-variation in conditional volatility at higher frequencies (on the order of $0.58^{1/30} = 0.98$ daily, say).

F Welfare costs calculations

Consider our model with N market types. Let $s_t = (g_t, \sigma_t)$ denote the realization of the aggregate state and let $s_{m,t} = (s_t, \hat{y}_{m,t})$ denote the realization of the state in market m. The lifetime utility of a representative trader in market $m \in \{1, \ldots, N\}$ is $y_0^{1-\gamma} \hat{v}_m(s_{m,0})$, where $\hat{v}_m(s_m)$ solves:

$$\hat{v}_m(s_m) = \frac{\hat{c}_m(s_m)^{1-\gamma}}{1-\gamma} + \mathbb{E}_{s_m}\left[\beta g(s')^{1-\gamma} \hat{v}(s'_m)\right],$$

where $\hat{c}_m(s_m)$ denotes the ratio of consumption to aggregate endowment in market m and state s_m, s'_m denotes the state next period, $g(s')$ denotes aggregate growth in state s', and $\mathbb{E}_{s_m}[\cdot]$ denotes expectations conditional on state s_m.

If there is no segmentation, then in every market m the lifetime utility is that of the Lucas (1987) representative agent, $y_0^{1-\gamma} \hat{v}_{lucas}(s)$, where

$$\hat{v}_{lucas}(s) = \frac{1}{1-\gamma} + \mathbb{E}_s\left[\beta g(s')^{1-\gamma} \hat{v}_{lucas}(s')\right].$$

Of course, $\hat{v}_{lucas}(s)$ depends on $s = (g, \sigma)$ only through aggregate consumption growth g. We now calculate the benefit of eliminating all segmentation, expressed as the percentage increase Ω in lifetime consumption that would make the family indifferent between living with the segmented markets or moving to the full-risk sharing allocation. As is familiar from Alvarez and Jermann (2004), given homogeneous utility functions, the welfare cost Ω solves:

$$(1+\Omega)^{1-\gamma}\, \mathbb{E}\left[\sum_{m=1}^{N} \omega_m \hat{v}_m(s_m)\right] = \mathbb{E}\left[\hat{v}_{lucas}(s)\right]$$

so that

$$\Omega = \left(\frac{\mathbb{E}\left[\hat{v}_{lucas}(s)\right]}{\mathbb{E}\left[\sum_{m=1}^{N} \omega_m \hat{v}_m(s_m)\right]}\right)^{\frac{1}{1-\gamma}} - 1. \tag{16}$$

To see the effects of segmentation in multiple markets, observe that we could alternatively calculate a market-specific cost of segmentation Ω_m such that:

$$(1+\Omega_m)^{1-\gamma}\mathbb{E}\left[\hat{v}(s_m)\right] = \mathbb{E}\left[\hat{v}_{lucas}(s)\right] \tag{17}$$

Plugging the expression for $\mathbb{E}\left[\hat{v}(s_m)\right]$ as a function of Ω_m into equation (16), we find that:

$$1+\Omega=\left[\sum_{m=1}^{N}\omega_m\left(1+\Omega_m\right)^{\gamma-1}\right]^{\frac{1}{\gamma-1}}.$$

So the aggregate cost Ω is a CES aggregate of the market specific costs Ω_m. In our calibration we have $\gamma=4$, so that $(1+\Omega_m)^{\gamma-1}$ is a *convex* function of Ω_m. By Jensen's inequality this implies that:

$$\Omega>\sum_{m=1}^{N}\omega_m\Omega_m.$$

However, in our numerical examples, the difference between the two turns out to be small.

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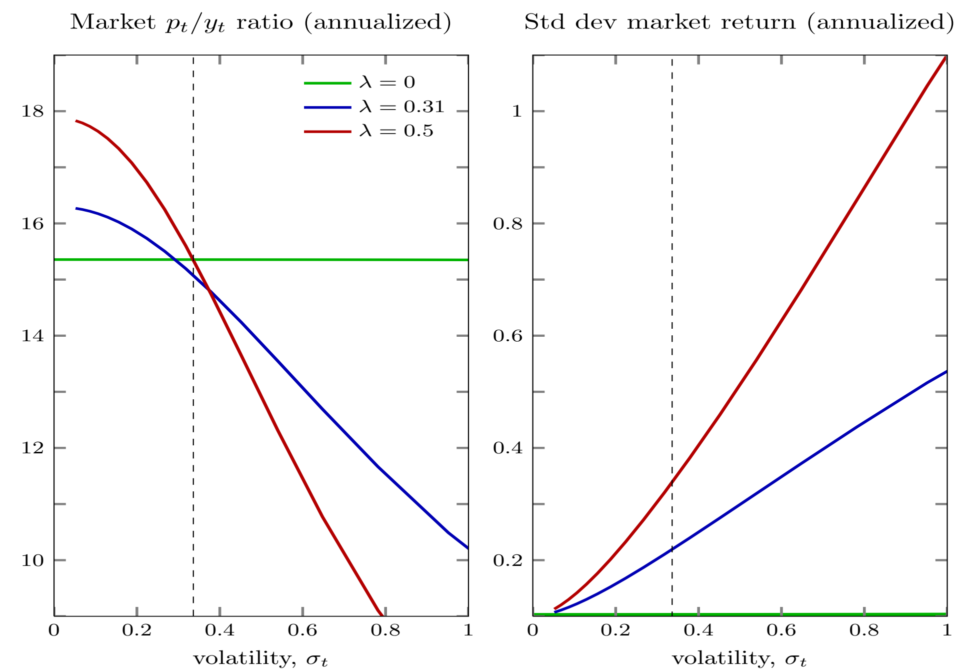
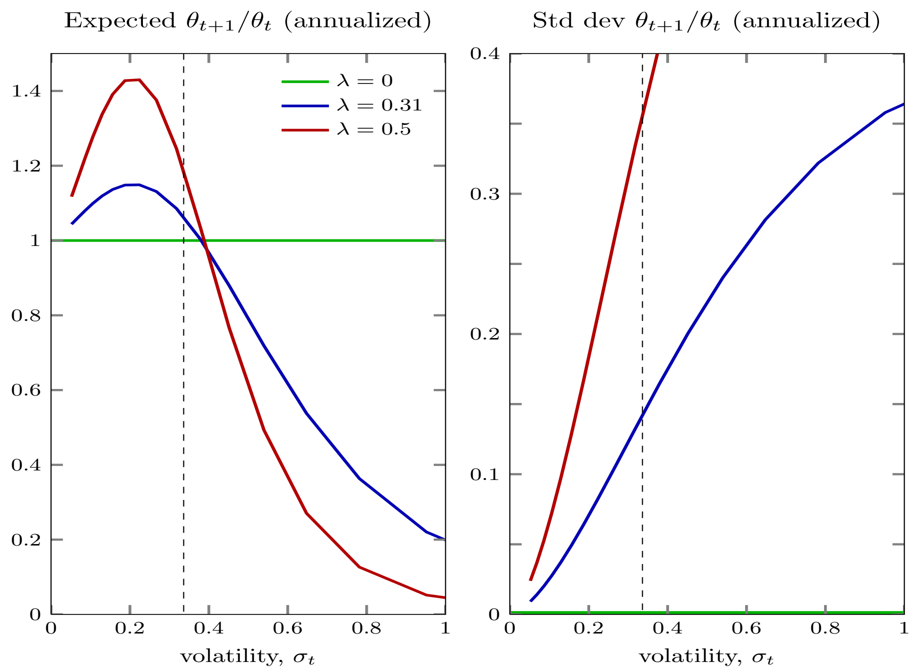
Figure I
Conditional moments.



The market p_t/y_t ratio (left panel) and conditional standard deviation of the market return (right panel), both as a function of the volatility state σ_t and expressed in annual terms. Three cases are shown, the frictionless case ($\lambda = 0$), our benchmark ($\lambda = 0.31$), and a high segmentation case ($\lambda = 0.5$). The aggregate endowment growth is fixed at its unconditional mean. The vertical dashed line is the unconditional mean $\bar{\sigma}$.

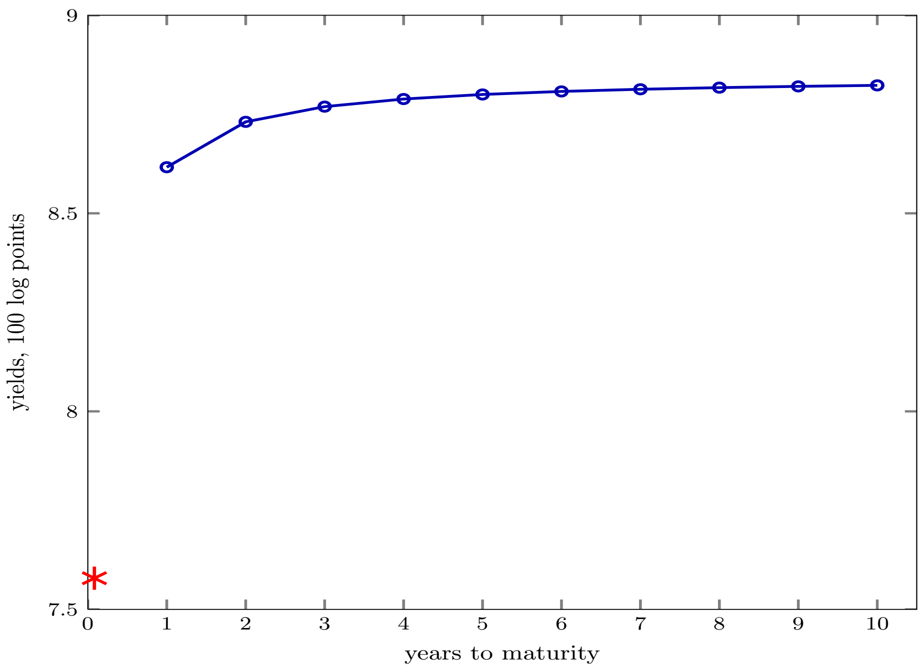
Figure II

Multiplicative bond-pricing factor θ_{t+1}/θ_t.



The expected aggregate bond-pricing factor θ_{t+1}/θ_t (left panel) and the standard deviation of θ_{t+1}/θ_t (right panel) as a function of the volatility state σ_t, all expressed in annual terms. Three cases are shown, the frictionless case ($\lambda = 0$), our benchmark ($\lambda = 0.31$), and a high segmentation case ($\lambda = 0.5$). The vertical dashed line is the unconditional mean $\bar{\sigma}$.

Figure III
Average yield curve (annualized).



Average yield curve for the benchmark model. The star point on the left is the average yield on a one-month zero coupon bond, $12\mathbb{E}\left[\log(R_f)\right]$. Note that, because the risk free rate is so volatile and because $\log(\cdot)$ is concave, this yield turns out to be about 1% lower than the average risk free rate ($R_f = 8.19\%$ annual) reported in the main text.

Table I

Aggregate statistics in annualized monthly data and in monthly data time-aggregated to yearly.

	Annualized monthly	Aggregated to yearly
average real risk-free rate	1.81	1.81
standard deviation of real risk free rate	1.20	2.43
average real NYSE return	7.24	7.27
standard deviation of real NYSE return	14.40	13.90
equity premium	5.43	5.47
standard deviation of equity premium	14.25	13.30
average price-dividend ratio	495.18	34.38
standard deviation of log price-dividend ratio	0.56	0.34
autocorrelation of log price-dividend ratio	−0.02	0.89
average consumption growth	2.19	2.17
standard deviation of consumption growth	1.25	1.33

Aggregate postwar US data. All return data is monthly 1959:1-2007:12 and reported in annualized percent. The stock market index is the value weighted NYSE return from CRSP, and the risk-free return is the 90 day T-bill rate. Real consumption growth refers to the growth of real nondurables and services consumption per capita from the BEA. The first column shows annualized statistics for monthly data. To annualize monthly returns and consumption growth, we multiply by 12, and to annualize monthly standard deviations, we multiply by $\sqrt{12}$. The second column shows statistics for yearly data, which are obtained by compounding returns and growth over the relevant time interval. The only statistics that are substantially different in this second column concern the price dividend ratio: this is because, in the first column the dividend that enters the ratio is the dividend per month, while in the second column it is the dividend paid over the entire year.

Table II

Robustness of single λ model solutions.

	Model							
	Benchmark			Constant		Feedback		
	Standard	High	No twist	Standard	High	Standard	High	No twist
Calibrated parameters								
λ	0.31	0.31	0.31	0.31	0.31	0.31	0.31	0.31
$\bar{\sigma}$	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32
$\sigma_{\epsilon v}$	0.21	0.21	0.21	0	0	0.21	0.21	0.21
ϕ	0.78	0.78	0.79	n/a	n/a	0.78	0.78	0.79
η	n/a	n/a	n/a	n/a	n/a	2.51	2.51	2.52
Fitted moments								
std dev diversified market portfolio return	4.16	4.16	4.16	1.01	1.01	4.16	4.16	4.16
average cross-section std dev returns	16.40	16.40	16.40	16.03	16.03	16.40	16.40	16.40
time-series std dev cross-section std dev returns	4.17	4.17	4.17	0	0	4.17	4.17	4.17
AR(1) cross-section std dev returns	0.84	0.84	0.84	n/a	n/a	0.84	0.84	0.84
cross-section std dev returns on lagged growth	n/a	n/a	n/a	n/a	n/a	−0.56	−0.56	−0.56
Asset pricing implications								
$\mathbb{E}[R_M - R_f]$	2.43	2.43	2.43	0.22	0.22	2.43	2.43	2.42
$\text{Std}[R_M - R_f]$	13.27	13.27	13.27	1.01	1.01	13.27	13.27	13.26
$\mathbb{E}[R_M - R_f]/\text{Std}[R_M - R_f]$	0.17	0.17	0.17	0.20	0.20	0.17	0.17	0.17
$\mathbb{E}[R_M]$	10.62	10.62	10.62	9.47	9.47	10.62	10.62	10.62
$\text{Std}[R_M]$	14.41	14.41	14.41	1.01	1.01	14.41	14.41	14.41
$\mathbb{E}[R_f]$	8.19	8.19	8.19	9.25	9.25	8.19	8.19	8.20
$\text{Std}[R_f]$	5.55	5.54	5.61	0	0	5.57	5.57	5.64
$\mathbb{E}[p/y]$	14.10	14.10	14.10	14.13	14.13	14.10	14.10	14.10
$\text{Std}[\log(p/y)]$	20.56	20.56	20.50	0	0	20.57	20.57	20.50
$\text{Auto}[\log(p/y)]$	0.76	0.76	0.76	n/a	n/a	0.76	0.76	0.76

The *standard* precision case has $N = N_g \times N_\sigma \times N_{\hat{y}} = 3 \times 9 \times 19 = 513$ quadrature nodes and weights. The *high* precision case has $N = N_g \times N_\sigma \times N_{\hat{y}} = 5 \times 19 \times 25 = 2,375$. In both these cases, the "twisted" density recommended by Flodèn (2008) is used to alleviate concerns about the accuracy of the Tauchen and Hussey (1991) procedure when the stochastic process is persistent (see equation (8) in Appendix B). The final *no twist* case uses the plain Tauchen and Hussey procedure and same configuration of nodes as in the standard case. The issue of twisting does not arise in the constant σ model.

Table III
Predictability regressions.

	Data					Model		
Regression	Coefficient	s.e.	R2	$\text{Std}[\mathbb{E}_t(R)]$	$\frac{\text{Std}[\mathbb{E}_t(R)]}{\mathbb{E}[R]}$	Coefficient	$\text{Std}[\mathbb{E}_t(R)]$	$\frac{\text{Std}[\mathbb{E}_t(R)]}{\mathbb{E}[R]}$
return on y/p								
one-year horizon	3.00	2.16	0.07	4.15	0.47	15.05	6.73	0.68
five-year horizon	17.52	3.19	0.27	4.49	0.52	4.54	10.14	0.17
excess return on y/p								
one-year horizon	3.42	2.52	0.09	4.73	0.63	−1.37	0.61	0.95
five-year horizon	17.44	2.54	0.27	4.06	0.59	−0.39	0.86	0.36

All return data is annual 1959–2007 and reported in percent. The stock market index is the value weighted NYSE return from CRSP, and the risk-free return is the 90 day T-bill rate. We obtain real returns after deflating by the CPI from the BLS. The regression standard errors use the Hansen and Hodrick (1980) correction for overlapping observations. The terms $\text{Std}[\mathbb{E}_t(R)]$ and $\text{Std}[\mathbb{E}_t(R)]/\mathbb{E}[R]$ are, respectively, the standard deviation and coefficient of variation of the fitted values of the regression.

Table IV
Properties of the multiplicative SDF adjustment factors.

Moment	Market-specific $\frac{\theta_{m,t+1}}{\theta_{m,t}}$	Aggregate $\frac{\theta_{t+1}}{\theta_t}$
expected value	1.014	1.001
standard deviation	0.173	0.046
autocorrelation	−0.461	−0.126
correlation with		
aggregate growth g_{t+1}	0.000	0.000
volatility σ_{t+1}	0.125	0.354
idiosyncratic growth $\hat{y}^m_{t+1}/\hat{y}^m_t$	−0.758	−0.056
$\mathrm{Std}[\mathbb{E}_t(\cdot)]$	0.116	0.015
$\mathrm{Auto}[\mathrm{Std}_t(\cdot)]$	0.576	0.778

The SDF that prices asset returns in market m is $\beta g_{t+1}^{-\gamma}\theta_{m,t+1}/\theta_{m,t}$ while the SDF pricing bonds is $\beta g_{t+1}^{-\gamma}\theta_{t+1}/\theta_t$. Each is the product of the standard Lucas-Breeden aggregate SDF $\beta g_{t+1}^{-\gamma}$ and a multiplicative adjustment factor. See Appendix E for details. The table reports the quantitative properties of these factors for our benchmark calibration. All statistics are monthly.